Tikrit university
College of Engineering
Mechanical Engineering Department

Lectures on Numerical Analysis

Chapter 4
Numerical Integration

Assistant prof. Dr. Eng. Ibrahim Thamer Nazzal

What does an integral represent?

$\int_a^b f(x)dx =$ area under the curve *f(x)* between $x = a$ to $x = b.$

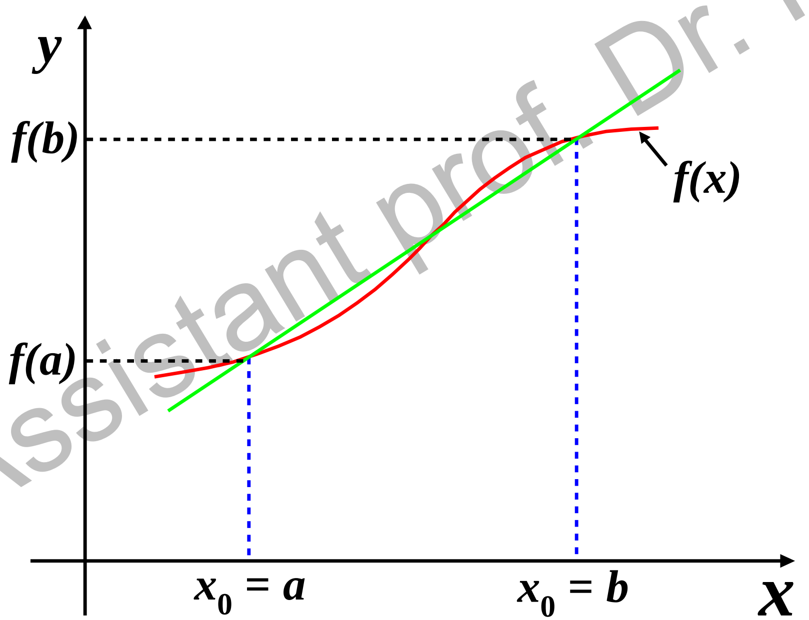
Where:

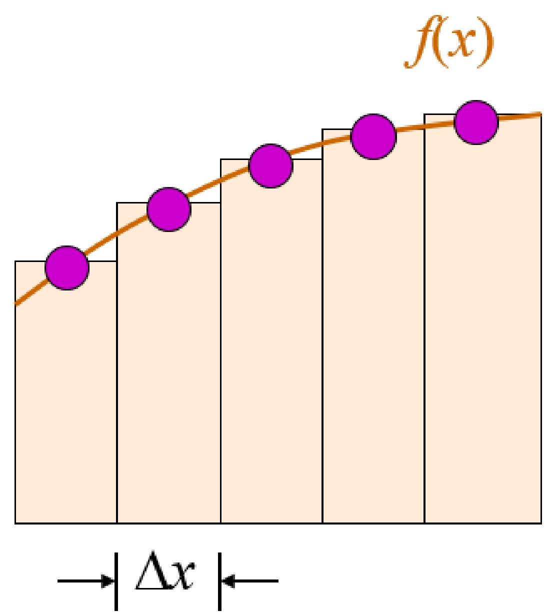
f(x) is the integrand a= lower limit of integration b= upper limit of integration

Basic definition of an integral:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x \qquad \Delta x = \frac{b-a}{n}$$

sum of height × width





- **Why numerical integration**
- **Evaluate the integral**

$$I = \int_a^b f(x)dx$$ **without doing the calculation analytically.**

In many cases a mathematical expression, For a complex non-linear continuous function that is difficult or impossible to integrate directly such as

$$f(x) = \frac{1 + x^3 + \sin x^2}{1 + \cos x}$$

$$f(x) = x^4 - \frac{1}{x} + x(\tan x - e^x)$$

$$f(x) = \frac{e^x}{3 + x^2} + x \ln x - \frac{\cos x}{x}$$

2. A tabulated continuous function where values of the independent variable x and $f(x)$ are given at a number of discrete data points as is often the case with experimental or field data such as distance traveled by a car vs. time:

t (sec)	0	2	4	6	8	10
D (ft)	0	10	50	150	330	610

The *Newton-Cotes formulas* are the most common numerical integration schemes.

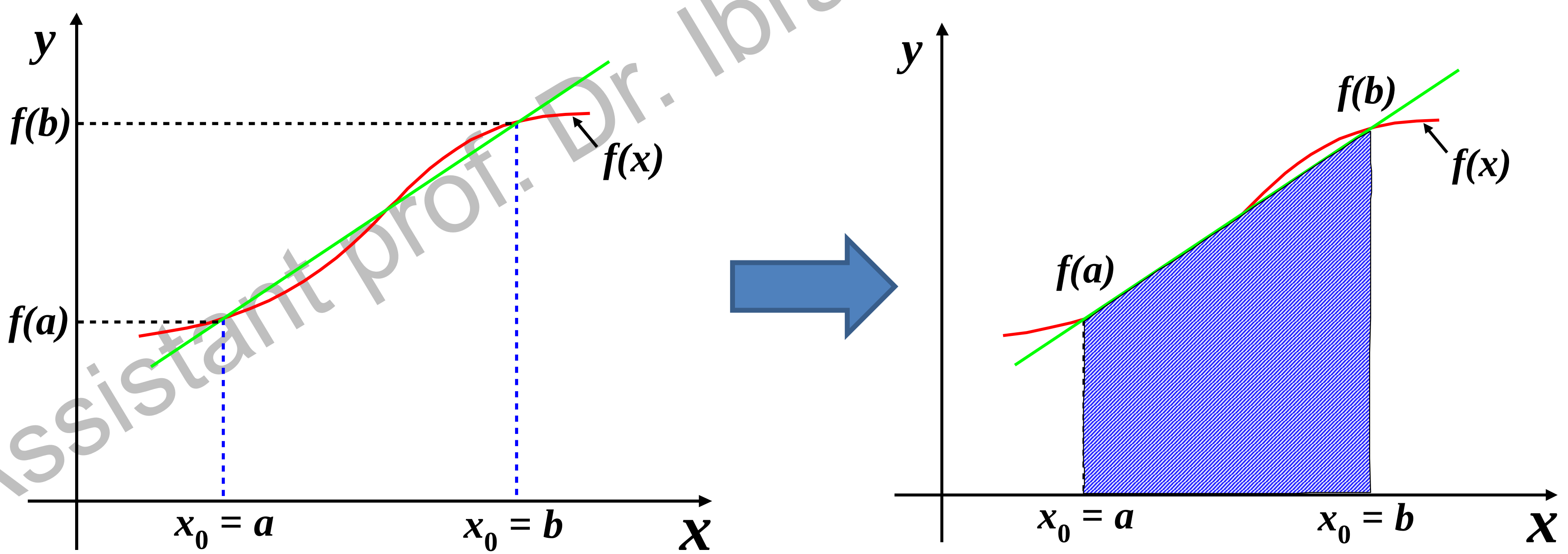
They are based on the strategy of replacing a complicated function or tabulated data with an approximating function that is easy to integrate:

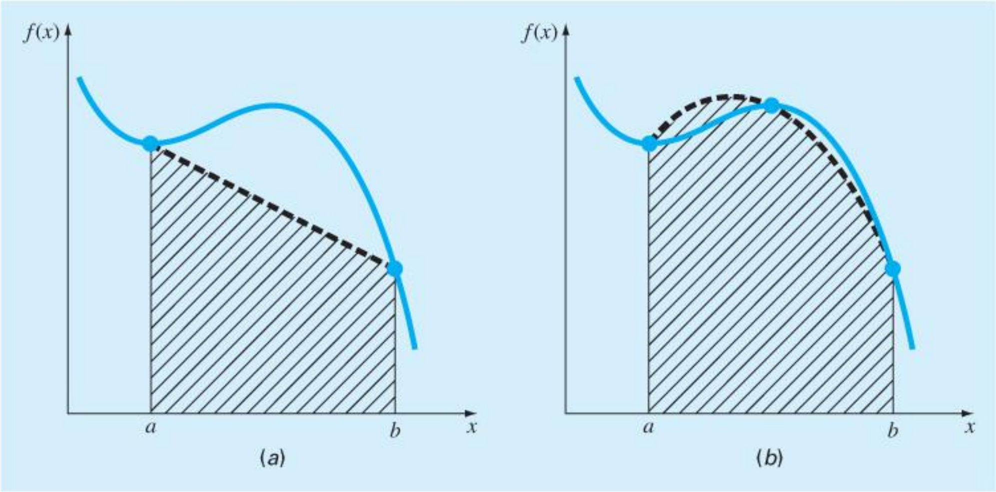
Newton-Cotes Methods/Formulae

The derivation of Newton-Cotes formula is based on Polynomial Interpolation.

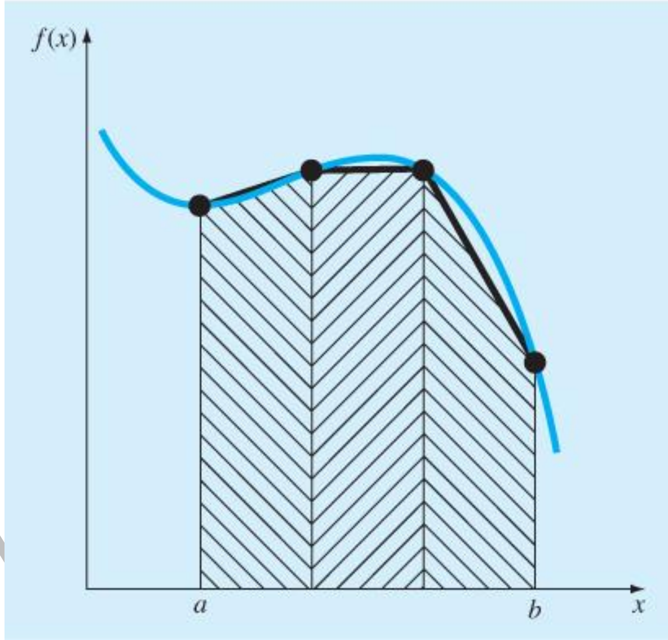
$$I = \int_a^b f(x)dx \quad \text{where} \quad f(x) \approx f_n(x)$$

$$f_n(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n$$





The integrating function can be polynomials for any order - for example, (a) straight lines or (b) parabolas



The integral can be approximated in one step or in a series of steps to improve accuracy

Types of Newton-Cotes Formulae

1) Trapezoidal Rule (**Two pint formula**)
2) Simpson's 1=3 Rule (**Three Point formula**)
3) Simpson's 3=8 Rule (**Four point formula**)

1. Trapezoidal Rule

The *Trapezoidal rule* is the first of the Newton-Cotes closed integration formulas, corresponding to the case where the polynomial is **first order:**

The trapezoidal rule can also be derived from geometry. Look at Figure The area under the curve *f(x)* is the area of a trapezoid. The integral

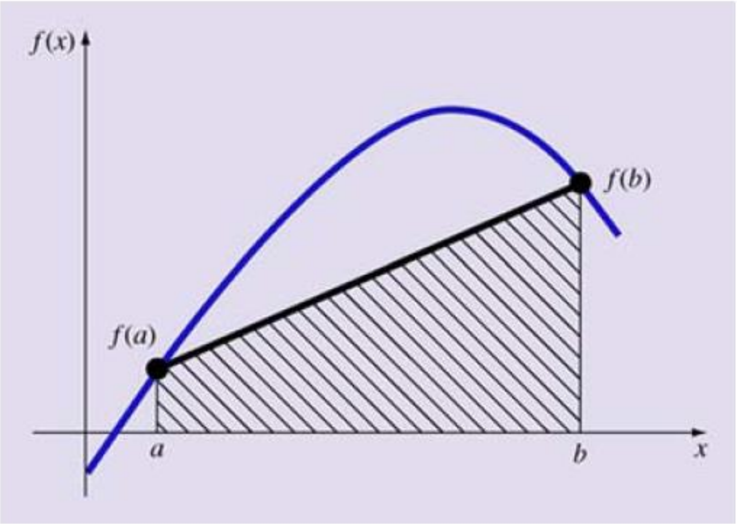
$$\int_a^b f(x)dx \approx \text{Area of trapezoid}$$

$$= \frac{1}{2} \times (\text{ Sum of length of parallel sides})(\text{Perpendicular distance between parallel sides})$$

$$= \frac{1}{2}(\textit{Sum of parallel sides})(\textit{height})$$

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$= \frac{h}{2}\left[f(a) + f(b)\right]$$



$$I = \text{width} \times \text{average height} = (b-a) \times \text{average height}$$

Derivation of the Trapezoidal Rule

We write our function under polynomial form:

$$\int_a^b f(x)dx \approx \text{Area of trapezoid}$$

$$= \text{area under curve}$$

$$f(x) \approx f_n(x)$$

where

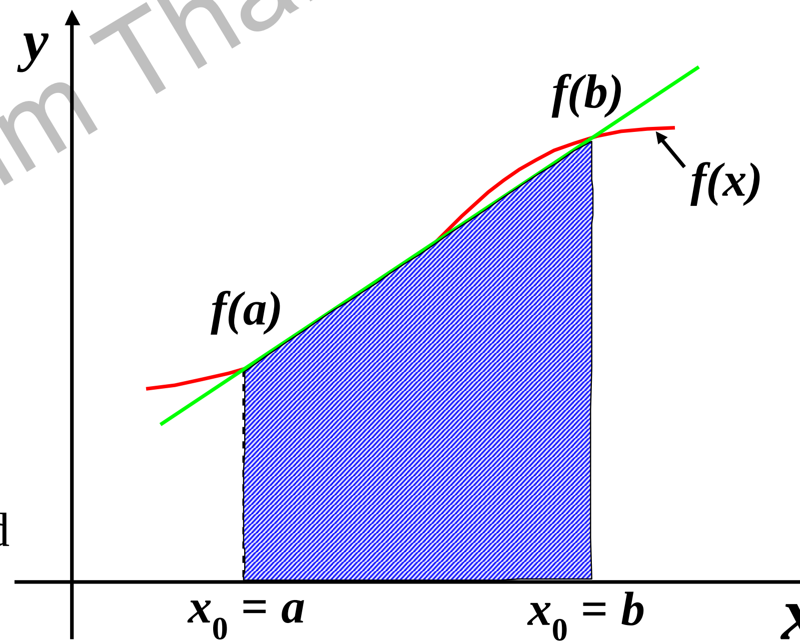
$$f_n(x) = a_0 + a_1 x + \dots + a_{n-1}x^{n-1} + a_n x^n$$

For trapezoidal rule, the area under this first order polynomial is an estimate of the integral of *f(x)* between the limits of *a* and *b*:

$$I = \int_a^b f(x)dx \cong \int_a^b f_1(x)dx$$

$$= \int_a^b (a_0 + a_1 x)dx$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right)$$



But what is a_0 and a_1. Now if we choose , $(a, f(a))$ and $(b, f(b))$ as the tow points to approximate $f(x)$by a straight line from a to b

$$f(a) = f_1(a) = a_0 + a_1 a$$

$$f(b) = f_1(b) = a_0 + a_1 b$$

Solving the above two equations for a and b

$$a_1 = \frac{f(b) - f(a)}{b-a} \qquad a_0 = \frac{f(a)b - f(b)a}{b-a}$$

Hence we get, $\int_a^b f(x)dx = \frac{f(a)b - f(b)a}{b-a}(b-a) + \frac{f(b)-f(a)}{b-a}\frac{b^2-a^2}{2}$

$$\int_a^b f(x)\ dx = (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

Example: Use the trapezoidal rule to numerically integrate $f(x) = 0{:}2 + 25x$ from $a = 0$ to $b = 2$.

Solution: $f(a) = f(0) = 0{:}2$, and $f(b) = f(2) = 50{:}2$

$$I = (b-a)\frac{f(b)+f(a)}{2} = (2-0)\times\frac{0.2+50.2}{2} = 50.4$$

The true solution

$$\int_0^2 f(x)dx = (0.2x + 12.5x^2)|_0^2 = (0.2\times 2 + 12.5\times 2^2) - 0 = 50.4$$

Because $f(x)$ is a linear function, using the trapezoidal rule gets the exact solution.

•Example 1: Use the trapezoidal rule to numerically integrate $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $a = 0$ to $b = 0.8$. Note that the exact value of the integral can be determined analytically to be 1.640533

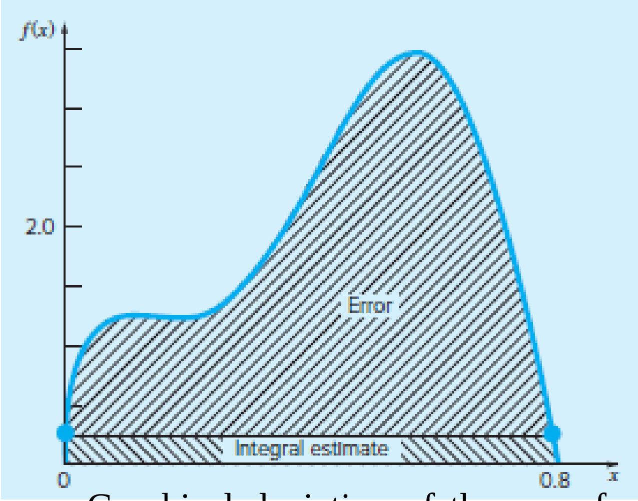
The true solution **1.640533**

Solution: $f(0) = 0.2$ and $f(0.8) = 0.232$

$$\int_a^b f(x)\ dx = (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

$$= (0.8-0)\ \frac{0.2+0.232}{2} = 0.1728$$

which represents an error of $\varepsilon_t = 1.640533 - 0.1728 = 1.467733$, which corresponds to a percent relative error of $\varepsilon_t = 89.5\%$. The reason for this large error is evident from the graphical depiction in Fig.. Notice that the area under the straight line neglects a significant portion of the integral lying above the line.



Graphical depiction of the use of a single application of the trapezoidal rule to approximate the integral of

Error of the Trapezoidal Rule

- **An estimate for the local truncation error of a single application of the trapezoidal rule is:**

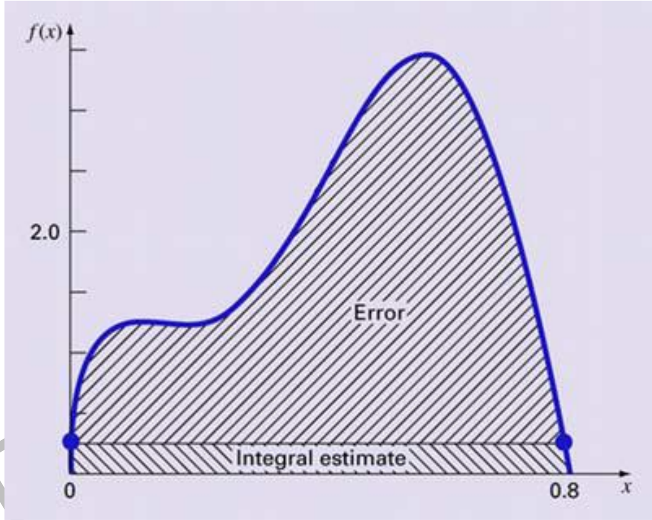
$$E_t = -\frac{1}{12} f''(\xi)(b-a)^3$$

where ξ is somewhere between a and b

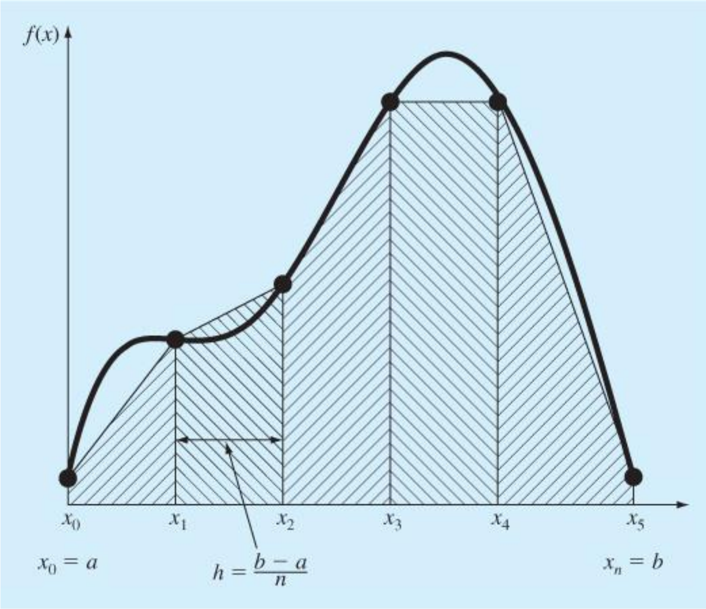
exact for the linear function

This formula indicates that the error is dependent upon the curvature of the actual function as well as the distance between the points

- **Error can thus be reduced by breaking the curve into parts**



Graphical depiction of the use of a single application of the trapezoidal rule to approximate the integral of from $x = 0$ to 0.8.



Example The vertical distance covered by a rocket from t=8 to t=30 seconds is given by:

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use single segment Trapezoidal rule to find the distance covered.

b) Find the true error, E_t for part (a).

c) Find the absolute relative true error, $|\in_a|$ for part (a).

Solution

a) $I \approx (b-a)\left[\frac{f(a)+f(b)}{2} \right]$ $a = 8$ $b = 30$

$$f(t) = 2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t$$

$$f(8) = 2000\, ln\left[\frac{140000}{140000 - 2100(8)} \right] - 9.8(8) = 177.27\ m/s$$

$$f(30) = 2000\, ln\left[\frac{140000}{140000 - 2100(30)} \right] - 9.8(30) = 901.67\ m/s$$

a) $I = (30-8)\left[\frac{177.27 + 901.67}{2} \right] = 11868\ m$

b) The exact value of the above integral is

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt = 11061\ m$$

b) $E_t = True\ Value - Approximate\ Value$

$= 11061 - 11868 = -807\ m$

c) The absolute relative true error, $|\in_t|$, would be

$$|\in_t| = \left|\frac{11061 - 11868}{11061}\right| \times 100 = 7.2959\%$$

Example

Use the trapezoidal rule to numerically integrate $f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $a = 0$ to $b = 0.8$. Note that the exact value is 1.640533.

Sol.)

$$I = (0.8 - 0)\frac{0.2 + 0.232}{2} = 0.1728 \quad \rightarrow \quad E_t = 1.640533 - 0.1728 = 1.467733 \quad \rightarrow \quad \varepsilon_t = 89.5\%$$

An approximate error estimate

$$f''(x) = -400 + 4{,}050x - 10{,}800x^2 + 8{,}000x^3$$

$$\bar{f}''(x) = \frac{\int_0^{0.8} (-400 + 4{,}050x - 10{,}800x^2 + 8{,}000x^3)dx}{0.8 - 0} = -60$$

$$E_a = -\frac{1}{12}(-60)(0.8)^3 = 2.56$$

The Multiple Application Trapezoidal Rule.

The composite trapezoidal rule using smaller integration interval can reduce the approximation error. We can divide the integration interval from a to b into a number of segments and apply the trapezoidal rule to each segment. Divide ($a; b$) into n segments of equal width. Then

$$h = \frac{b-a}{n} \qquad If\ a = x_0\ and\ b = x_n$$

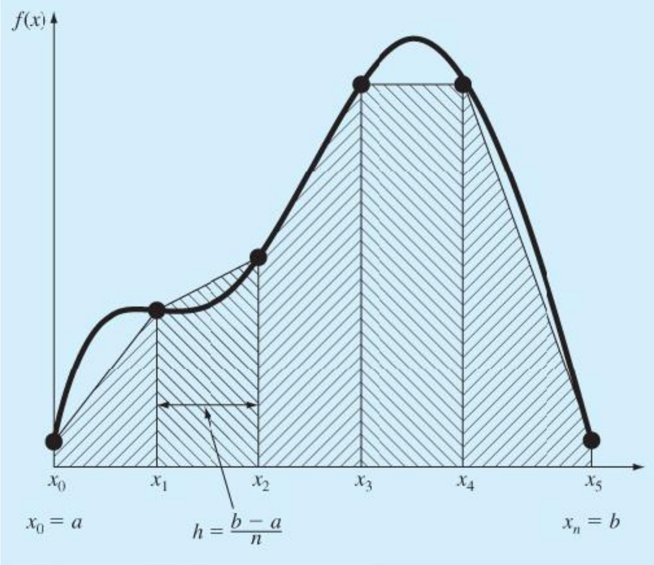
$$I = \int_{x_0}^{x_n} f_n(x)\,dx = \int_{x_0}^{x_1} f_n(x)\,dx + \int_{x_1}^{x_2} f_n(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f_n(x)\,dx$$

Substituting the trapezoidal rule

$$I = (x_1 - x_0)\frac{f(x_0)+f(x_1)}{2} + (x_2 - x_1)\frac{f(x_1)+f(x_2)}{2} + \cdots + (x_n - x_{n-1})\frac{f(x_{n-1})+f(x_n)}{2}$$

$$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

$$I = \underbrace{(b-a)}_{\text{Width}}\underbrace{\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}}_{\text{Average height}}$$



E_r an error for the composite trapezoidal rule (obtained by summing the individual errors for each segment)

$$E_t = -\frac{(b-a)^3}{12n^3}\sum_{i=1}^{n} f''(\xi_i)$$

Example The function is given in the following tabulated form. Compute $\int_0^1 f(x)dx$ with h= 0.25 and h= 0. 5 with, using the composite trapezoidal method.

x	0	0.25	0. 5	0.75	1
$f(x)$	0.9162	0.8109	0.6931	0.5596	0.4055

solution

For the given data, below equation can be used to integrate by the composite trapezoidal method:

$$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

for h= 0.25, there are four intervals, and N= 4, *the* equation gives

$$I(f) \approx \frac{0.25}{2}(0.9162 + 0.4055) + 0.25(0.8109 + 0.6931 + 0.5596)$$

$$I(f) \approx 0.6811$$

for h= 0.5, there are two intervals, and N= 2, *the* equation gives

$$I(f) \approx \frac{0.5}{2}(0.9162 + 0.4055) + 0.5(0.6931)$$

$$I(f) \approx 0.6770$$

Example

Use the Trapezoidal rule to estimate the integral $I = \int_0^1 \frac{dx}{1+x^2}$ taking $h = 1/4$ intervals.

Solution

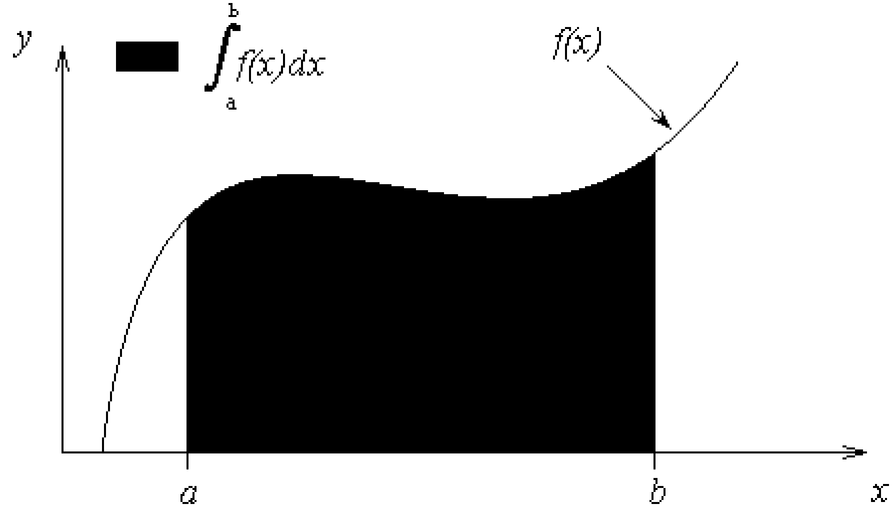
At first, we shall tabulate the function as

x	0	1/4	1/2	3/4	1
$\frac{1}{1+x^2}$	1	0.9412	0.8	0.64	0.5

using the Trapezoidal rule, and taking $h = 1/4$

$$I = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)] = \frac{1}{8}[1.5 + 2(2.312)] = 0.7828$$

Trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial in the interval of integration. Simpson's 1/3rd rule is an extension of Trapezoidal rule where the integrand is non-approximated by a second order polynomial.



Deriving Simpson's Rule

Hence

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

where $f_2(x)$ is a second order polynomial.

$$f_2(x) = a_0 + a_1 x + a_2 x^2$$

Choose $(a, f(a)), \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right),$ and $(b, f(b))$

as the three points of the function to evaluate a_0, a_1 and a_2

$$f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2$$

Solving the above three equations for unknowns, a_0, a_1 and a_2 give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Then

$$I \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b$$

$$= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}$$

Substituting values of a_0, a_1 and a_2

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since for Simpson's 1/3rd Rule, the interval [a, b] is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

Hence the Simpson's 1/3rd rule is given by

$$\int_a^b f(x)dx \cong \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since the above form has 1/3 in its formula, it is also called Simpson's 1/3rd Rule.

Example 1

The distance covered by a rocket from t=8 to t=30 is given by

$$x = \int_{8}^{30} \left(2000\,ln\left[\frac{140000}{140000-2100t}\right] - 9.8t \right) dt$$

a) Use Simpson's 1/3rd Rule to find the approximate value of x

b) Find the true error, ϵ_t

c) Find the absolute relative true error, $|\epsilon_t|$

Solution

a) $x = \int_{8}^{30} f(t)dt$ $\quad x = \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

$a = 8$ $\quad b = 30$ $\quad \frac{a+b}{2} = 19$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t$$

$$f(8) = 2000\ln\left[\frac{140000}{140000-2100(8)}\right] - 9.8(8) = 177.27 m/s$$

$$f(30) = 2000\ln\left[\frac{140000}{140000-2100(30)}\right] - 9.8(30) = 901.67 m/s$$

$$f(19) = 2000\ln\left(\frac{140000}{140000-2100(19)}\right) - 9.8(19) = 484.75 m/s$$

$$x \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

$$= \left(\frac{30-8}{6}\right)\left[f(8) + 4f(19) + f(30) \right]$$

$$= \left(\frac{22}{6}\right)\left[177.2667 + 4(484.7455) + 901.6740\right] = 11065.72 \text{ m}$$

b) The exact value of the above integral is

$$x = \int_{8}^{30}\left(2000\, ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t \right) dt = 11061.34\ m$$

$$E_t = True\ Value - Approximate\ Value \qquad E_t = 11061.34 - 11065.72 = -4.38\ m$$

c) Absolute relative true error,

$$\left|\in_t\right| = \left|\frac{11061.34 - 11065.72}{11061.34}\right| \times 100\% = 0.0396\%$$

Multiple Segment Simpson's 1/3rd Rule

Just like in multiple-segment Trapezoidal Rule, one can subdivide the interval *[a, b]* into *n* segments and apply Simpson's 1/3rd Rule repeatedly over every two segments. Note that n needs to be even. Divide interval *[a,b] into n* equal segments, hence the segment width

$$h = \frac{b-a}{n}$$

$$\int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx \quad \text{where} \quad x_0 = a \quad x_n = b$$

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots\ldots + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$

Apply Simpson's 1/3rd Rule over each interval,

$$\int_a^b f(x)dx \cong (x_2 - x_0)\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right] + (x_4 - x_2)\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right] + \ldots$$

$$+ (x_{n-2} - x_{n-4})\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right] + (x_n - x_{n-2})\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

Since $x_i - x_{i-2} = 2h \quad i = 2, 4, \ldots, n$

$$\int_a^b f(x)dx \cong 2h\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right] + 2h\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right] + \ldots$$

$$+ 2h\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right] + 2h\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\left\{ f(x_1) + f(x_3) + \ldots + f(x_{n-1}) \right\} + 2\left\{ f(x_2) + f(x_4) + \ldots + f(x_{n-2}) \right\} + f(x_n) \right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n) \right]$$

$$\int_a^b f(x)dx \cong \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n) \right]$$

Example 2

Use 4-segment Simpson's 1/3 rule to approximate the distance covered by a rocket in meters from t = 8 s to t= 30 s as given by

$$x = \int_8^{30} \left(2000 \ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use Simpson's 1/3rd Rule to find the approximate value of x

b) Find the true error, ϵ_t part (a).

c) Find the absolute relative true error, $|\epsilon_t|$part (a).

Solution:

a) Using n segment Simpson's 1/3 rule,

$$x \approx \frac{b-a}{3n}\left[f(t_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(t_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(t_i) + f(t_n) \right]$$

$$n = 4 \quad a = 8 \qquad b = 30$$

$$h = \frac{b-a}{n} = \frac{30-8}{4} = 5.5$$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t$$

So $f(t_0) = f(8)$

$$f(8) = 2000\ln\left[\frac{140000}{140000-2100(8)}\right] - 9.8(8) = 177.27 m/s$$

$$f(t_1) = f(8+5.5) = f(13.5)$$

$$f(13.5) = 2000\ln\left[\frac{140000}{140000-2100(13.5)}\right] - 9.8(13.5) = 320.25\,m/s$$

$$f(t_2) = f(13.5+5.5) = f(19)$$

$$f(19) = 2000\ln\left(\frac{140000}{140000-2100(19)}\right) - 9.8(19) = 484.75\,m/s$$

$$f(t_3) = f(19+5.5) = f(24.5)$$

$$f(24.5) = 2000\ln\left[\frac{140000}{140000-2100(24.5)}\right] - 9.8(24.5) = 676.05\,m/s$$

$$f(t_4) = f(t_n) = f(30)$$

$$f(30) = 2000\ln\left[\frac{140000}{140000-2100(30)}\right] - 9.8(30) = 901.67\,m/s$$

$$x = \frac{b-a}{3n}\left[f(t_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(t_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(t_i) + f(t_n) \right]$$

$$= \frac{30-8}{3(4)}\left[f(8) + 4\sum_{\substack{i=1 \\ i=odd}}^{3} f(t_i) + 2\sum_{\substack{i=2 \\ i=even}}^{2} f(t_i) + f(30) \right]$$

$$= \frac{22}{12}\left[f(8) + 4f(t_1) + 4f(t_3) + 2f(t_2) + f(30) \right]$$

$$= \frac{11}{6}\left[f(8) + 4f(13.5) + 4f(24.5) + 2f(19) + f(30) \right]$$

$$= \frac{11}{6}\left[177.27 + 4(320.25) + 4(676.05) + 2(484.75) + 901.67\right] = 11061.64\ m$$

b) The exact value of the above integral is

$$x = \int_{8}^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t\right)dt \quad = 11061.34 \text{ m}$$

So the true error is

$$E_t = True\ Value - Approximate\ Value$$

$$E_t = 11061.34 - 11061.64 \quad = -0.30\ m$$

c) The absolute relative true error is

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100 = \left|\frac{-0.3}{11061.34}\right| \times 100 = 0.0027\%$$

Table 1 Values of Simpson's 1/3 rule for Example 2 with multiple-segments

n	Approximate Value	E_t	$\lvert\in_t\rvert$
2	11065.72	-4.38	0.0396%
4	11061.64	-0.30	0.0027%
6	11061.40	-0.06	0.0005%
8	11061.35	-0.02	0.0002%
10	11061.34	-0.01	0.0001%

Examples Find Solution using composite Simpson's 1/3 rule

x	1.4	1.6	1.8	2	2.2
y	4.0552	4.953	6.0436	7.3891	9.025

Solution:

Using Simpsons 1/3 Rule

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n) \right]$$

$$\int f(x)dx = \frac{h}{3}[(y_0+y_4) + 4(y_1 + y_3) + 2(y_2)]$$

$$\int y\, dx = \frac{h}{3}[(y_0+y_4) + 4(y_1 + y_3) + 2(y_2)]$$

$$\int y\, dx = \frac{0.2}{3}[(4.0552 + 9.025) + 4 \times (4.953 + 7.3891) + 2 \times (6.0436)]$$

$$\int y\, d = \frac{0.2}{3}[(4.0552 + 9.025) + 4 \times (12.3421) + 2 \times (6.0436)]$$

$$= 4.9691$$

Error in Multiple Segment Simpson's 1/3rd Rule

The true error in a single application of Simpson's 1/3rd Rule is given by

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

In Multiple Segment Simpson's 1/3rd Rule, the error is the sum of the errors in each application of Simpson's 1/3rd Rule. The error in n segment Simpson's 1/3rd Rule is given by

$$E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2 \qquad = -\frac{h^5}{90} f^{(4)}(\zeta_1)$$

$$E_2 = -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4 \qquad = -\frac{h^5}{90} f^{(4)}(\zeta_2)$$

$$E_{\frac{n}{2}} = -\frac{(x_n - x_{n-2})^5}{2880} f^{4}\left(\zeta_{\frac{n}{2}}\right), \quad x_{n-2} < \zeta_{\frac{n}{2}} < x_n \qquad = -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}}\right)$$

Hence, the total error in Multiple Segment Simpson's 1/3rd Rule is

:

$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i \quad = -\frac{h^5}{90}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) \quad = -\frac{(b-a)^5}{90n^5}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) \quad = -\frac{(b-a)^5}{90n^4}\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

The term $\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$ is an approximate average value of $f^{(4)}(x),\ a < x < b$

Hence

$$E_t = -\frac{(b-a)^5}{90n^4}\overline{f}^{(4)}$$

where

$$\overline{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

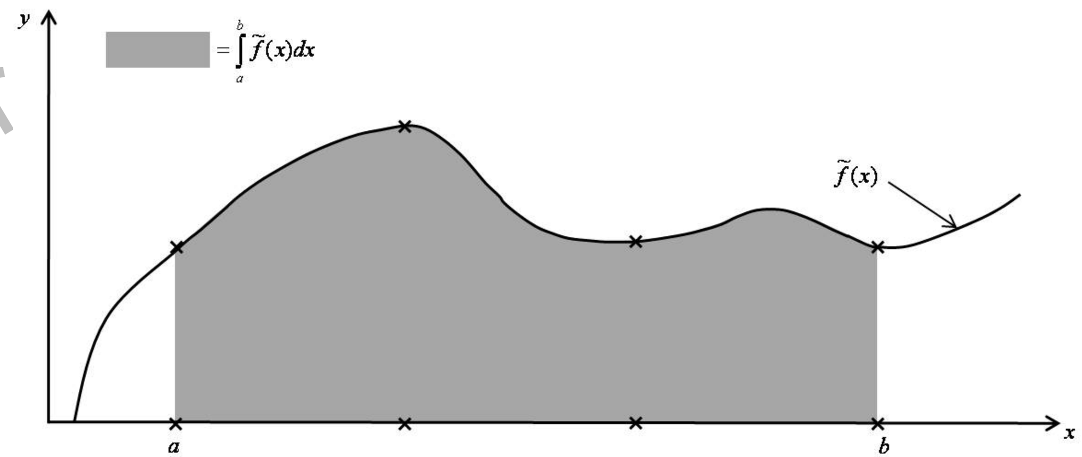
Introduction

The main objective of this section is to develop appropriate formulas for approximating the integral of the form

$$I = \int_a^b f(x)dx$$

Most (if not all) of the developed formulas for integration are based on a simple concept of approximating a given function by a simpler function (usually a polynomial function) , where represents the order of the polynomial function. In Chapter, Simpsons 1/3 rule for integration was derived by approximating the integrand with a 2nd order (quadratic) polynomial function. $f(x)$ $fi(xi\ )$

Previously, it has been explained and illustrated that Simpsons 1/3 rule for integration can be derived by replacing the given function with the 2nd –order (or quadratic) polynomial function, defined as:



$$f_i(x) = f_2(x)$$

(2)

$$f_2(x) = a_0 + a_1 x + a_2 x^2$$

In a similar fashion, Simpson 1/3 rule for integration can be derived by replacing the given function with the 3rd-order (or cubic) polynomial (passing through 4 known data points) function defined as $f_i(x) = f_3(x)$

$$\left.\begin{aligned} f_3(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ &= \{1, x, x^2, x^3\} \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \end{aligned}\right\} \qquad (3)$$

which can also be symbolically represented in Figure 1.

The unknown coefficients a_0, a_1, a_2 *and* a_3 (in Eq. (3)) can be obtained by substituting 4 known coordinate data points $\{x_0, f(x_0)\}, \{x_1, f(x_1)\}, \{x_2, f(x_2)\}$ *and* $\{x_3, f(x_3)\}$ into Eq. (3), as following

$$\left.\begin{aligned} f(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 + a_3 x_0^2 \\ f(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^2 \\ f(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^2 \\ f(x_3) &= a_0 + a_1 x_3 + a_2 x_3^2 + a_3 x_3^2 \end{aligned}\right\} \qquad (4)$$

Equation (4) can be expressed in matrix notation as

Eq. (4) can be expressed in matrix notation as

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{Bmatrix} \qquad (5)$$

The above Eq. (5) can be symbolically represented as

$$[A]_{4\times4}\vec{a}_{4\times1} = \vec{f}_{4\times1} \qquad (6)$$

Thus,

$$\vec{a} = \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = [A]^{-1} \times \vec{f} \qquad (7)$$

Substituting Eq. (7) into Eq. (3), one gets

$$f_3(x) = \{1, x, x^2, x^3\} \times [A]^{-1} \times \vec{f} \qquad (8)$$

Remarks

As indicated in Figure 1, one has

$$\left.\begin{aligned} x_0 &= a \\ x_1 &= a + h = a + \frac{b-a}{3} = \frac{2a+b}{3} \\ x_2 &= a + 2h = a + \frac{2b-2a}{3} = \frac{a+2b}{3} \\ x_3 &= a + 3h = a + \frac{3b-3a}{3} = b \end{aligned}\right\} \quad (9)$$

With the help from MATLAB [2], the unknown vector $\vec{a}$ (shown in Eq. 7) can be solved.

Thus, Eq. (1) can be calculated as (See Eqs. 8, 10 for Method 1 and Method 2, respectively):

(11)

Simpsons 3/8 Rule for Integration

Substituting the form of $f_3(x)$ from Method (1) or Method (2),

$$I = \int_a^b f(x)dx$$

$$\approx \int_a^b f_3(x)dx$$

$$I = \int_a^b f(x)dx \approx \int_a^b f_3(x)dx$$

$$I = (b-a) \times \frac{\{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\}}{8} \qquad (11)$$

Since $h = \frac{b-a}{3}$ $\qquad b - a = 3h$

And equation 11 becomes:

$$I \approx \frac{3h}{8} \times \{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\} \qquad (12)$$

The error introduced by the Simpson 3/8 rule can be derived as:

$$E_t = -\frac{(b-a)^5}{6480} \times f''''(\zeta)$$, where $$a \le \zeta \le b \qquad (13)$$

Example 2

Use Simpson 3/8 rule to approximate the distance covered by a rocket in meters from t = 8 s to t= 30 s as given by

$$x = \int_{8}^{30} \left(2000 \ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

Use Simpson 3/8 rule to find the approximate value of the integral.

Solution

$$h = \frac{b-a}{3} = \frac{30-8}{3} = 7.3333$$

$$x_0 = 8 \Rightarrow f(x_0) = 2000 \ln\left(\frac{140000}{140000 - 2100 \times 8} \right) - 9.8 \times 8 = 177.2667$$

$$\begin{cases} x_1 = x_0 + h = 8 + 7.3333 = 15.3333 \\ f(x_1) = 2000 \ln\left(\dfrac{140000}{140000 - 2100 \times 15.3333} \right) - 9.8 \times 15.3333 = 372.4629 \end{cases}$$

$$\begin{cases} x_2 = x_0 + 2h = 8 + 2(7.3333) = 22.6666 \\ f(x_2) = 2000 \ln\left(\dfrac{140000}{140000 - 2100 \times 22.6666} \right) - 9.8 \times 22.6666 = 608.8976 \end{cases}$$

$$\begin{cases} x_3 = x_0 + 3h = 8 + 3(7.3333) = 30 \\ f(x_3) = 2000\ln\left(\dfrac{140000}{140000 - 2100 \times 30}\right) - 9.8 \times 30 = 901.6740 \end{cases}$$

Applying Eq. (12), one has:

$$I = \frac{3}{8} \times 7.3333 \times \{177.2667 + 3 \times 372.4629 + 3 \times 608.8976 + 901.6740\}$$

$$I = 11063.3104$$

The “exact” answer can be computed as

$$I_{exact} = 11061.34$$

Multiple Segments for Simpson 3/8 Rule

Using n = number of equal (small) segments, the width h can be defined as

$$h = \frac{b-a}{3} \qquad (14)$$

The number of segments need to be an integer multiple of 3 as a single application of Simpson 3/8 rule requires 3 segments.

The integral shown in Equation (1) can be expressed as

$$I = \int_a^b f(x)dx \approx \int_a^b f_3(x)dx$$

$$I \approx \int_{x_0=a}^{x_3} f_3(x)dx + \int_{x_3}^{x_6} f_3(x)dx + \ldots\ldots\ldots + \int_{x_{n-3}}^{x_n=b} f_3(x)dx \qquad (15)$$

Using Simpson 3/8 rule (See Equation 12) into Equation (15), one gets

$$I = \frac{3h}{8}\left\{ \begin{array}{l} f(x_0)+3f(x_1)+3f(x_2)+f(x_3)+f(x_3)+3f(x_4)+3f(x_5)+f(x_6) \\ +\ldots..+f(x_{n-3})+3f(x_{n-2})+3f(x_{n-1})+f(x_n) \end{array} \right\} \qquad (16)$$

$$= \frac{3h}{8}\left\{ f(x_0)+3\sum_{i=1,4,7,..}^{n-2} f(x_i)+3\sum_{i=2,5,8,..}^{n-1} f(x_i)+2\sum_{i=3,6,9,..}^{n-3} f(x_i)+f(x_n) \right\} \qquad (17)$$

Example 2 (Multiple segments Simpson 3/8 rule) The vertical distance in meters covered by a rocket from t= 8 to t = 30 seconds is given by

$$I = \int_{a=8}^{b=30} \left\{ 2000 \ln\left(\frac{140{,}000}{140{,}000 - 2100x} \right) - 9.8x \right\} dx,$$

using Simple 3/8 multiple segments rule, with number six segments to estimate the vertical distance .

Solution

In this example, one has (see Eq. 14):

$$f(t) = 2000 \ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t$$

$$h = \frac{30-8}{6} = 3.6666$$

$$\{t_0, f(t_0)\} = \{8, 177.2667\}$$

$$\{t_1, f(t_1)\} = \{11.6666, 270.4104\} \text{ where } t_1 = t_0 + h = 8 + 3.6666 = 11.6666$$

$$\{t_2, f(t_2)\} = \{15.3333, 372.4629\} \text{ where } t_2 = t_0 + 2h = 15.3333$$

$$\{t_3, f(t_3)\} = \{19, 484.7455\} \text{ where } t_3 = t_0 + 3h = 19$$

$$\{t_4, f(t_4)\} = \{22.6666, 608.8976\} \text{ where } t_4 = t_0 + 4h = 22.6666$$

$$\{t_5, f(t_5)\} = \{26.3333, 746.9870\} \text{ where } t_5 = t_0 + 5h = 26.3333$$

$$\{t_6, f(t_6)\} = \{30, 901.6740\} \text{ where } t_6 = t_0 + 6h = 30$$

Applying Eq. (17), one obtains:

$$I = \frac{3}{8}(3.6666)\left\{177.2667 + 3\sum_{i=1,4,..}^{n-2=4} f(t_i) + 3\sum_{i=2,5,..}^{n-1=5} f(t_i) + 2\sum_{i=3,6,..}^{n-3=3} f(t_i) + 901.6740\right\}$$

$$I = (1.3750)\left\{\begin{matrix}177.2667 + 3(270.4104 + 608.8976) + 3(372.4629 + 746.9870) \\ + 2(484.7455) + 901.6740\end{matrix}\right\}$$

$$= 11{,}601.4696$$

(b) The number of multiple segments that can be used in the conjunction with Simpson 1/3 rule is 2,4,6,8,.. (any even numbers).

$$I_1 = \left(\frac{h}{3}\right)\{f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + f(x_4) + ... + f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\}$$

$$I_2 = \left(\frac{h}{3}\right)\left\{f(x_0) + 4\sum_{i=1,3,\ldots}^{n-1} f(x_i) + 2\sum_{i=2,4,6\ldots}^{n-2} f(x_i) + f(x_n)\right\}$$

However, Simpson 3/8 rule can be used with the number of segments equal to 3,6,9,12,.. (can be either certain odd or even numbers).

(c) If the user wishes to use, say 7 segments, then the mixed Simpson 1/3 rule (for the first 4 segments), and Simpson 3/8 rule (for the last 3 segments).

Remarks:

(a) Comparing the truncated error of Simpson 1/3 rule

$$E_t = -\frac{(b-a)^5}{2880} \times f''''(\zeta) \qquad (18)$$

With Simple 3/8 rule (See Eq. 13), the latter seems to offer slightly more accurate answer than the former. However, the cost associated with Simpson 3/8 rule (using 3rd order polynomial function) is significant higher than the one associated with Simpson 1/3 rule (using 2nd order polynomial function).